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Quantum Jeffreys prior for displaced squeezed thermal states

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Abstract. It is known that, by extending the equivalence of the Fisher information matrix to its quantum version, the Bures metric, the quantum Jeffreys prior can be determined from the volume element of the Bures metric. We compute the Bures metric for the displaced squeezed thermal state and analyse the quantum Jeffreys prior and its marginal probability distributions. To normalize the marginal probability density function, it is necessary to provide a range of values of the squeezing parameter or the inverse temperature. We find that if the range of the squeezing parameter is kept narrow, there are significant differences in the marginal probability density functions in terms of the squeezing parameters for the displaced and undisplaced situations. However, these differences disappear as the range increases. Furthermore, marginal probability density functions against temperature are very different in the two cases.

The relentlessly rapid miniaturization of integrated circuits invariably implies a need to explore computation at the atomic scale. It is therefore natural to amalgamate two seemingly unrelated disciplines, namely classical computational theories and quantum mechanics. Indeed, the recent progress with remarkable breakthroughs in quantum algorithms [1] and quantum hardwares [2] have immensely enhanced the possibility of realizing a quantum computer.

When one combines classical information theories with quantum theories, one begins to see the potential application of this amalgamation for the transmission and processing of information. However, unlike classical theories, repeated sampling of quantum systems are not always permitted and one needs to consider carefully the observer's ability to select an optimal strategy for a given set of signals whose prior probabilities are known [3, 4]. Indeed, one such strategy involves the minimization of the loss of information or maximization of mutual information by reducing Shannon entropy for an ensemble of signals. In fact, given *a priori* information concerning the nature of the signals, one can seek a strategy by replacing the prior distributions by posterior distributions in accordance with the data from the observations. This process is possible through the formalism of the Bayesian inferential paradigm.

Related to this choice of strategy but starting from a very different approach is the geometrical investigation of the quantum analogue of the Fisher information in classical statistics. The quantum analogue turns out to be related to the Bures metric [5–8]. More specifically, the Bures metric allows the experimenter to distinguish infinitesimally close density operators.

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An ensemble of quantum states can be defined as a collection of normalized states $|\psi_1\rangle, \ldots |\psi_n\rangle$ with fixed *a priori* probabilities $p_1, \ldots, p_n$, respectively [9]. Associated with this ensemble, one can define its density matrix ρ as

$$\rho = \sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i|. \tag{1}$$

For pure states, this density matrix can always be diagonalized into a matrix with only one non-zero eigenvalue. To distinguish between pure states $|\psi_1\rangle$ and $|\psi_2\rangle$, one considers the Fubini–Study distance D_{FS} and obtains its minimum value, that is

$$\begin{aligned} D_{\mathrm{FS}}^2(|\psi_1\rangle, |\psi_2\rangle) &= \inf \| \, |\psi_1\rangle - \mathrm{e}^{\mathrm{i}\theta}|\psi_2\rangle\|^2 \\ &= 2(1 - |\langle\psi_1|\psi_2\rangle|) \end{aligned} \tag{2}$$

where θ is the relative phase between the states, $|\psi_1\rangle$ and $|\psi_2\rangle$.

For mixed states, one needs to use density operator formalism. One can extremize the Hilbert–Schmidt metric, the extension of the Fubini–Study metric, to an enlarged Hilbert space and show that the equivalent distance for distinguishing two density matrices, ρ_1 and ρ_2, on a Hilbert space is the Bures distance given by

$$D_{\mathrm{B}}^2(\rho_1, \rho_2) = 2\left(1 - \mathrm{tr}\sqrt{\rho_1^{1/2}\rho_2\rho_1^{1/2}}\right). \tag{3}$$

To derive the Bures metric, $\mathrm{d}s_{\mathrm{B}}^2$, one can introduce a real parameter, t, and consider the perturbative expansion in t to second order so that the metric becomes

$$d_{\mathrm{B}}^2 = g_{ij}(\rho)\,\mathrm{d}\rho^i\,\mathrm{d}\rho^j \tag{4}$$

$$= \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} D_{\mathrm{B}}(\rho, \rho + d\rho)^2|_{t=0} \tag{5}$$

where ρ^i and ρ^j are canonical coordinates of the density operators in the manifold and $\mathrm{d}\rho$ is an infinitesimal change in the density operator. Furthermore, Einstein's summation convention is implicitly assumed.

The investigation of the Bures metric for pure states and its geometrical properties have been widely discussed [7, 10, 11] and the results have also been extended to impure or mixed states for spin-$\frac{1}{2}$ systems in which the density matrix can be expressed succinctly as

$$\rho = \tfrac{1}{2}(1 + n_1\sigma_x + n_2\sigma_y + n_3\sigma_z) \tag{6}$$

where σ_i are Pauli matrices and n_i are the components of the density matrix in the Bloch sphere, $i = 1 \ldots 3$. However, as noted by Twamley [6], fewer explicit results have been obtained for other mixed states. Twamley has, therefore, recently computed an explicit expression for the Bures metric for the squeezed thermal states. Nevertheless, Twamley did not discuss the situation for the displaced squeezed states due to the difficulty of finding a faithful matrix representation for the displaced exponential operators. Recently, Paraoanu and Scutaru [12,13] have explicitly worked out the Bures distance for displaced thermal states. However, it is difficult to obtain the exact Bures metric for the displaced squeezed thermal states on the basis of their results. By considering the Baker–Campbell–Hausdorff (BCH) formula for quadratic operators [14], we have computed an explicit expression for the Bures distance [15] in terms of the parameters in the density matrices.

In this paper, we build on our recent results [15] concerning the Bures distance for the displaced squeezed states. We consider the density operator for the displaced squeezed thermal state

$$\rho = Z(\beta) D S \Lambda S^\dagger D^\dagger \tag{7}$$

where

$$D = \exp\left[(a^\dagger, a)\begin{pmatrix} k \\ -k^* \end{pmatrix}\right] \qquad \text{and} \qquad S = \exp[\tfrac{1}{2}r((a^2 - a^{\dagger 2})]$$

are unitary operators. Furthermore, in equation (7), the operator Λ and the normalization factor $Z(\beta)$ are given respectively by $\exp[-\frac{\beta}{2}(aa^\dagger + a^\dagger a)]$ and $(\mathrm{tr}\,\Lambda)^{-1}$ where β is the inverse temperature. The dagger symbol (†) in equation (7) denotes the Hermitian conjugate. We have also considered the squeezing parameter, r, as a real number.

Using equation (3) and the BCH relation [14, 16],

$$\begin{aligned} &S(a^\dagger, a)S^\dagger = (a^\dagger, a)M \qquad S^\dagger(a^\dagger, a)S = (a^\dagger, a)M^{-1} \\ &\Lambda(a^\dagger, a)\Lambda^{-1} = (a^\dagger, a)B \end{aligned} \tag{8}$$

where

$$M = \begin{pmatrix} \cosh r & -\sinh r \\ -\sinh r & \cosh r \end{pmatrix} \qquad \text{and} \qquad B \equiv \begin{pmatrix} \exp(-\beta) & 0 \\ 0 & \exp(\beta) \end{pmatrix}.$$

In our previous paper, we first computed the quantity $(\mathrm{tr}\sqrt{\rho_1{}^{\frac{1}{2}}\rho_2\rho_1{}^{\frac{1}{2}}})^2$ called the Bures fidelity. The explicit result [15] for the Bures distance, using equation (3), is

$$D_{\mathrm{B}}^2 = 2\left(1 - \exp\left\{\frac{1}{\Delta}(\epsilon_1 + \epsilon_2)\right\}\frac{2\sinh\frac{\beta_1}{4}\sinh\frac{\beta_2}{4}}{\sqrt{Y} - 1}\right) \tag{9}$$

where $Y = \cosh^2(r_1 - r_2)\cosh^2\frac{\beta_1+\beta_2}{4} - \sinh^2(r_1 - r_2)\cosh^2\frac{\beta_1-\beta_2}{4}$, $\Delta = \cosh\beta_1\cosh\beta_2 + \sinh\beta_1\sinh\beta_2\cosh 2(r_1 - r_2) - 1$ and

$$\epsilon_1 = \sinh\beta_1\sinh^2\frac{\beta_2}{2}[(p^2 - q^2)\sinh 2r_1 - 2(p^2 + q^2)\cosh 2r_1] \tag{10}$$

$$\epsilon_2 = \sinh^2\frac{\beta_1}{2}\sinh\beta_2[(p^2 - q^2)\sinh 2r_2 - 2(p^2 + q^2)\cosh 2r_2]. \tag{11}$$

Here q and p denote the canonical position and momentum respectively. Note that the parameters k and k^* have been absorbed into the position and momentum using the relations

$$p = \frac{1}{\sqrt{2}}(k + k^*) \tag{12a}$$

$$q = \frac{1}{\mathrm{i}\sqrt{2}}(k - k^*). \tag{12b}$$

In particular, we recover Twamley's result for the squeezed thermal states in the limit when $k = k^* = 0$ and the result for the displaced thermal states obtained by Paraoanu and Scutaru for $r = 0$.

To compute the Bures metric, we apply equation (5) to the Bures distance in equation (9). A straightforward calculation yields

$$\begin{aligned} d_{\mathrm{B}}^2 = {}& \frac{1}{2}\tanh\frac{\beta}{2}(2\cosh(2r) - \sinh(2r))\,\mathrm{d}p^2 + \frac{1}{2}\tanh\frac{\beta}{2}(2\cosh(2r) + \sinh(2r))\,\mathrm{d}q^2 \\ & + \frac{1}{2}\left[1 + \mathrm{sech}\frac{\beta}{2}\right]\mathrm{d}r^2 + \frac{1}{64\sinh^2\beta/4}\,\mathrm{d}\beta^2. \end{aligned} \tag{13}$$

By generalizing Wootters' formulation [17] of a statistical distance between quantum states, Braunstein and Caves [5] have shown that, up to a factor, the Bures distance for density matrices is equivalent to the Fisher information matrix. To be more specific, if we consider N

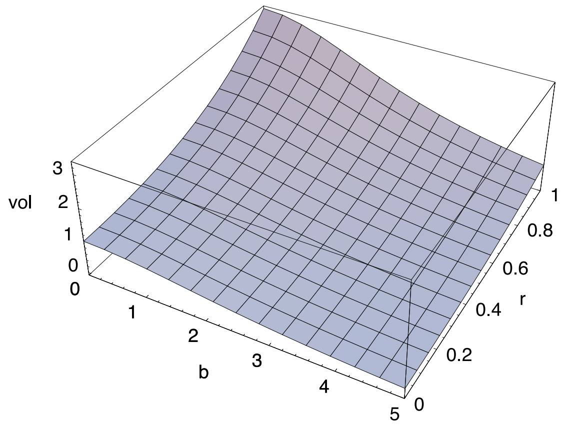

Figure 1. Variation of the volume element against the temperature parameter, β, and the squeezing parameter, r.

measurements, $\xi_1, \xi_2, \ldots$ and estimate the parameter θ using the function $\theta = \theta(\xi_1, \xi_2, \ldots)$, then according to Braunstein and Caves, the distinguishability metric can be defined as

$$\mathrm{d}s^2 \equiv \frac{\mathrm{d}\theta^2}{\min[N\langle(\delta\theta)^2\rangle]}. \tag{14}$$

On the other hand, one knows from Bayesian probabilistic theories [18] that the non-informative Jeffreys prior† is proportional to the square root of the determinant of the Fisher information matrix. Furthermore, the Fisher information matrix is determined by the Riemannian metric on the probability phase space and is therefore invariant to reparametrization. Relying on the similarities between the classical and quantum analysis, Slater [19, 20] has extended the classical Bayesian theory to its quantum version by defining the quantum prior probability distribution as proportional to the square root of the determinant of the Bures metric. Indeed, Slater [20] has already considered the quantum Jeffreys prior for the squeezed thermal states obtained by Twamley [6].

For displaced squeezed states, since the Bures metric is diagonal, we can easily compute the associated volume element, $\mathrm{d}V$, and the result is

$$\begin{aligned} \mathrm{d}V &= \left(\frac{1}{2}\cosh^2\frac{\beta}{4}\operatorname{sech}^{3/2}\frac{\beta}{2}\right)\sqrt{4\cosh^2(2r) - \sin^2(2r)}\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r\,\mathrm{d}\beta \\ &\equiv f(\beta)g(r)\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r\,\mathrm{d}\beta. \end{aligned} \tag{15}$$

We have plotted the variation of this volume element in figure 1 against the parameters β and r.

Since the quantum Jeffreys prior is proportional to the volume element and the volume element $\mathrm{d}V$ can be factorized as a product of univariate function, following Slater [20], we can indeed consider the univariate marginal probability distributions for β and r by considering the functions $f(\beta)$ and $g(r)$ separately. In figure 2, we plot the univariate marginal probability distribution for $f(\beta)$ and compared the distribution with the results of the undisplaced squeezed thermal state. It is not possible to obtain an exact form for the normalization factor and we have

† In Bayesian theories, one distinguishes between *a priori* probability, or prior probability, and *a posteriori* probability, or posterior probability. Thus, if $\{E_j, j \in J\}$ (J is an arbitrary index set) are exclusive and exhaustive events and D is some given data, then one defines $p(H_j)$ as the prior probabilities and $p(H_j|D)$ as the posterior probabilities and $p(D|H_j)$ as the likelihood of the events H_j.

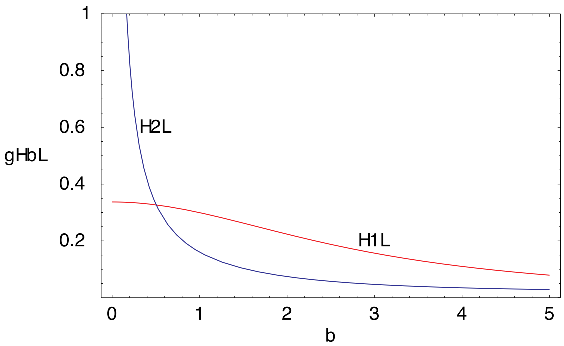

Figure 2. Marginal probability distribution of quantum Jeffreys prior for the temperature parameter, β with $\beta_0 = 5$. Curve (1) refers to the distribution for the displaced squeezed state and curve (2) refers to the distribution for the undisplaced squeezed state. Note that whereas the marginal distribution for the undisplaced squeezed states (curve 2) tends to infinity as $\beta \rightarrow 0$ (high temperature), the distribution for the displaced squeezed states (curve 1) goes to a finite value in the same limit.

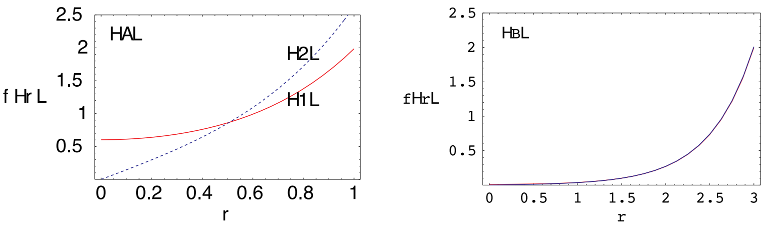

Figure 3. Plot A and plot B show the marginal probability distribution of quantum Jeffreys prior for the squeezing parameter, r, with $R = 1$ and $R = 3$, respectively. As in figure 2, for plot A, curve (1) refers to the distribution for displaced squeezed state and curve (2) refers to the distribution for the undisplaced squeezed state.

computed the marginal probability distribution for the temperature parameter β numerically over the range $0 \leqslant \beta \leqslant \beta_0$. Note that, whereas the marginal probability distribution for the undisplaced squeezed state diverges as $\beta \rightarrow 0$ or at high temperature, in the case of the displaced squeezed state, the marginal probability distribution goes to a finite value. The result is reminiscent of a similar situation in chi-square distribution curves in which the probability density function diverges at one degree of freedom, but not with higher degrees of freedom. This analogy seems to indicate that the change in the marginal probability density function in terms of inverse temperature stems from an increased degree of freedom associated with the displacement of the squeezed states.

In figure 3, we have considered the marginal probability for the squeezing parameter. For the displaced squeezed states, the normalization factor for the distribution, defined over the interval, $0 \leqslant r \leqslant R$, is an elliptic integral of the second kind, specifically the value is $-\mathrm{i}E(2\mathrm{i}R, \frac{3}{4})$, where $E(\phi, m)$ is the elliptic integral of the second kind with parameters ϕ and m. It is interesting to note that the marginal probability distributions over r differ significantly for small R values, but as R becomes larger, the two graphs coincide. Mathematically, it is not hard to understand why. The ratio

$$\frac{\sqrt{4\cosh^2(2r) - \sinh^2(2r)}}{\sinh(2r)}$$

and the ratio of the normalization factors

$$\frac{-\mathrm{i}E(2R\mathrm{i}, 3/4)}{\sinh^2[R]}$$

approach the same constant value of $\sqrt{3}$ as r goes to infinity. Physically, this seems to indicate that if the squeezing parameter can vary over a wider range, there is essentially no difference in the prior probability distribution of the squeeze parameter for displaced or undisplaced squeezing states.

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